CALCULATION OF LIMITING CURRENT OF RELATIVISTIC ELECTRON BEAMS

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Evacuated or gas-filled waveguides are generally used to transport high-current electron beams. An important characteristic of such a system is the limiting current of relativistic electrons which can be transmitted through a drift space without charge compensation (the limiting vacuum current). The standard method of calculating the limiting vacuum current is based on an analysis of the solution of the electrostatic problem far from the injecting plane. Smith and Hartman [1] used this method to find the limiting current for a nonrelativistic beam; Bogdankevich and Rukhadze [2] generalized this result for a relativistic beam and derived an interpolation formula for the limiting current of the form

$$J_* = \frac{mc^3}{e}\frac{1}{1+2\ln\left(\frac{R}{a}\right)}(\gamma_0^{2/3}-1)^{3/2}, \tag{1}$$

where m and e are the rest mass and charge of the electron, R and a are, respectively, the radii of the waveguide and beam, γ_0 is the beam energy in units of mc^2, and c is the speed of light.

In view of the fact that Eq. (1) was derived by completely neglecting the effect of the injection plane on the potential distribution in the drift space, there is a question of whether the theoretical limiting vacuum current can actually be attained (cf., e.g., [3]). In principle a virtual cathode (a singularity in the potential distribution) may be formed near the injection plane, and this can lead to cutoff of the beam current for values smaller than that given by Eq. (1). In other words, a special proof is required that the limiting current predicted from an analysis of the one-dimensional electrostatic problem far from the injection plane can actually be attained. Ryutov [3] gave such a proof for an infinitely narrow beam. It is clearly of interest to investigate the quasistatic potential distribution in a two-dimensional transport system (waveguide), taking account of the injection plane, in order to predict the limiting current actually attainable. We have performed such an investigation and have found the dependence of the limiting vacuum current on the beam parameters in a two-dimensional system.

Supposing a beam with current J_b passes along the axis of a waveguide located in an external magnetic field H_0, which is strong enough to ensure the complete magnetization of the electron beam

$$H_0^2/8\pi \gg n_b mc^2\gamma_0,$$

where n_b is the electron density in the beam. For beam currents smaller than the limiting value the electrons undergo single-current motion which is described by the following system of equations:

$$\Delta\varphi = 4\pi e n_e(r, z),\ mc^2(\gamma-\gamma_0) = e\varphi(r, z),$$
$$n_e(r, z)\, v_e(r, z) = n_b v_b,$$

where $n_e(r, z)$ and $v_e(r, z)$, the density and velocity of the electron beam, depend on the coordinates: v_b is the velocity of the injected beam.

Eliminating v_e and n_e from the system of equations, and transforming to the dimensionless variables

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$$x \equiv r/R,\ y \equiv z/R,\ \Phi \equiv e\varphi/mc^2,\ \alpha \equiv a/R,\ I_b \equiv (4e/mc^3)J_b,$$

we obtain the following nonlinear equation for the dimensionless potential Φ:

$$\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\Phi}{\partial x}\right)+\frac{\partial^2\Phi}{\partial y^2}=\frac{I_b}{\alpha^2}\frac{(\gamma_0+\Phi)\,\Theta(\alpha-x)}{\sqrt{(\gamma_0+\Phi)^2-1}}, \tag{2}$$

where

$$\Theta(x)=\begin{cases}0, & x<0,\\ 1, & x\geqslant 0.\end{cases}$$

By iterating with respect to the parameter I_b, the solution of Eq. (2) can be written in the form

$$\begin{aligned}&\Phi_0=0,\\ &\Phi_1=\int dx'dy'G(x,x';\ y,y')\,\Pi[\Phi_0(x',y')],\\ &\ldots\\ &\Phi_n=\int dx'dy'G(x,x';\ y,y')\,\Pi[\Phi_{n-1}(x',y')],\end{aligned} \tag{3}$$

where $G(x, x'; y, y')$ is the Green's function for Eq. (2), and the function $\Pi(\Phi)$ is given by

$$\Pi[\Phi(x,y)]=\frac{I_b}{\alpha^2}\frac{[\gamma_0+\Phi(x,y)]\,\Theta(\alpha-x)}{\sqrt{[\gamma_0+\Phi(x,y)]^2-1}}.$$

As shown by Kucherov [4], such a solution is valid up to the limiting values of the beam current given by Eq. (1). Therefore, the limiting current can be found from (3) by equating the potential on the axis to the kinetic energy of the beam. We note that the situation actually turns out to be more subtle: When the beam current reaches the limiting value a flux of reflected electrons appears. In this case the single-current Eq. (2) becomes invalid, because its right-hand side becomes infinite, and the iterative series (3) diverges. Therefore, the limiting current can be defined as the maximum for which a solution of the form (3) is still valid.

We show that the proposed definition of the limiting current is equivalent to that introduced earlier in [1-3], and find the limiting currents for a cylindrical wavelength with and without taking account of the injection plane.

In the simplest case, when the electron beam travels between the infinite plane electrodes, the solution of Eq. (2) can be reduced to quadratures. This approximation corresponds to the motion of electrons at an infinite distance from the injection plane in a gap of width 2d between two infinite coaxial cylinders as the radius of the inner cylinder $R_1 \to \infty$. Then by replacing x by $x_1 + R_1/d$, Eq. (2) takes the form (everywhere from now on $\alpha = 1$)

$$\frac{d^2\Phi}{dx^2}=I_b\frac{\gamma_0+\Phi}{\sqrt{(\gamma_0+\Phi)^2-1}},\quad \Phi(1)=\Phi'(0)=0.$$

The solution of this equation determining the relation between the potential Φ at $\bar{x} = 0$ ($\Phi_* = \Phi(0)$) and the current has the form

$$\int_0^{\Phi_*}\frac{d\Phi}{\sqrt{\sqrt{(\gamma_0+\Phi)^2-1}-\sqrt{(\gamma_0+\Phi_*)^2-1}}}=-\sqrt{2I_b}.$$

Figure 1 shows the dependence of the potential Φ_* on the beam current I_b for various values of the beam energy γ_0 (1 — $\gamma_0 = 2$; 2 — $\gamma_0 = 3$; 3 — $\gamma_0 = 5$). Although this relation is two-valued, only the lower branches of the curves in Fig. 1 have physical meaning. The limiting current in this case is defined as the current corresponding to $dI_b/d\Phi_* = 0$ [1, 3].

An iterative series of type (3) in the present case is given by

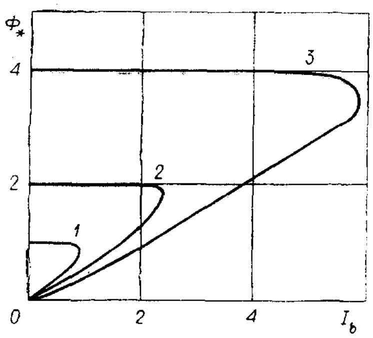

Fig. 1

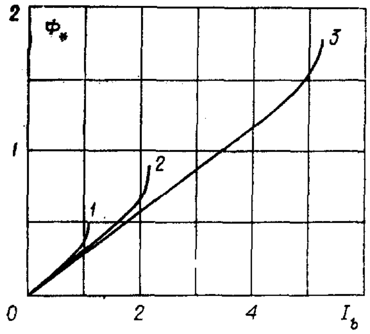

Fig. 2

$$\Phi_0(x) = 0,$$
$$\dots \tag{4}$$
$$\Phi_n(x) = I_b \int_0^1 G(x, x') \Pi[\Phi_{n-1}(x')]\, dx',$$

where

$$G(x, x') = \begin{cases} 1 - x', & 0 \leqslant x \leqslant x', \\ 1 - x, & x' \leqslant x \leqslant 1. \end{cases}$$

By summing the iterative series (4) numerically a relation can be obtained between the potential and the beam current and used to calculate values of the limiting currents. The values obtained agree with the analytic results shown in Fig. 1 to within the limits of accuracy of the machine calculations.* This indicates that the limiting currents can be calculated accurately enough by our iterative process.

We investigate first the behavior of the potential far from the injection plane, and find the dependence of the limiting current on the beam energy. In this approximation Eq. (2) reduces to the form

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d\Phi}{dx}\right) = I_b \frac{\gamma_0 + \Phi}{\sqrt{(\gamma_0 + \Phi)^2 - 1}}. \tag{5}$$

This equation can be solved by numerical summation of the iterative series (4), where

$$G(x, x') = \begin{cases} \ln x', & 0 \leqslant x \leqslant x', \\ \ln x, & x' \leqslant x \leqslant 1. \end{cases} \tag{6}$$

The values of the potential Φ_* calculated by this method for points on the beam axis far from the injection plane are plotted in Fig. 2 as functions of the beam current I_b for a cylindrical waveguide (1, γ_0 = 1.5; 2, γ_0 = 2; 3, γ_0 = 3).

The limiting current I_* for a given γ_0 is determined from the condition $dI_b/d\Phi_* = 0$. We note that the approximate relation (1) can be obtained from the iterative solution of (5). As the zero approximation we take $\Phi_0(x) = \Phi_*$. Then in the first approximation the potential distribution has the form

$$\Phi_1(x) = I_b \int_0^1 x'dx' G(x, x') \frac{\gamma_0 + \Phi_*}{\sqrt{(\gamma_0 + \Phi_*)^2 - 1}}.$$

Performing the integration, setting x = 0, and expressing I_b in terms of the remaining parameters, we obtain

*The calculations were limited to a relative accuracy of ~ 1%, since on the one hand this accuracy is quite satisfactory for comparison with approximate formulas and experiments, and on the other hand does not require a large amount of machine time.

$$I_b = -4\Phi_* \sqrt{1 - 1/(\gamma_0 + \Phi_*)^2}.$$

From the last expression the limiting current in the first approximation is

$$I_{*1} = 4\left(\gamma_0^{2/3} - 1\right)^{3/2}, \tag{7}$$

which in dimensional variables agrees with Eq. (1).

Figure 3 shows the dependence of the limiting current on the beam energy. Curve 2 corresponds to Eq. (7); curve 1 was obtained from the iterative solution (6). A comparison of these shows that the absolute error of Eq. (7) increases with increasing beam energy. At the same time the relative error $(I_* - I_{*1})/I_*$ in the range $1 < \gamma_0 < 5$ is $\approx$ 10-20%, and increases as $\gamma_0 \to 1$. The approximate expression for curve 1 is given by

$$I_* = 4\left(\gamma_0^{2/3} - 1\right)^{3/2}\left(1 + \frac{1}{4\gamma_0^{1/4}}\right) \tag{8}$$

with an accuracy of 3% for values of the energy in the range $1 < \gamma_0 < 20$.

Taking account of the injection plane requires using Eq. (2) without simplifying assumptions. For a semiinfinite cylindrical waveguide the boundary conditions for Eq. (2) are

$$\Phi_x'(0, y) = \Phi(1, y) = 0, \quad \Phi(x, 0) = \Phi_y'(x, \infty) = 0.$$

We seek the solution of Eq. (2) in the form of the expansion

$$\Phi(x, y) = \sum_{k=1}^{\infty} J_0(\lambda_k x)\, U_k(y),$$

where J_0 is the zero-order Bessel function ($J_0(\lambda_k) = 0$).

Then for the function $U_k(y)$ we obtain the equation

$$\frac{d^2U_k}{dy^2} - \lambda_k^2 U_k = \frac{2I_b}{J_1^2(\lambda_k)} \int_0^1 dx' x' J_0(\lambda_k x')\, \Pi[\Phi(x', y)].$$

The Green's function for this equation has the form

$$G(y, y') = -\frac{1}{2\lambda_k}\left(e^{-\lambda_k|y-y'|} - e^{-\lambda_k|y+y'|}\right).$$

The potential $\Phi(x, y)$ is given by the iterative series

$$\Phi_0(x, y) = 0,$$

$$\Phi_1(x, y) = 2I_b \sum_{k=1}^{\infty} \frac{J_0(\lambda_k x)}{J_1^2(\lambda_k)} \int_0^{\infty} G(y, y')\, dy' \int_0^1 x' dx' J_0(\lambda_k x')\, \Pi(\Phi_0),$$

$$\ldots$$

$$\Phi_n(x, y) = 2I_b \sum_{k=1}^{\infty} \frac{J_0(\lambda_k x)}{J_1^2(\lambda_k)} \int_0^{\infty} G(y, y')\, dy' \int_0^1 x' dx' J_0(\lambda_k x')\, \Pi(\Phi_{n-1}).$$

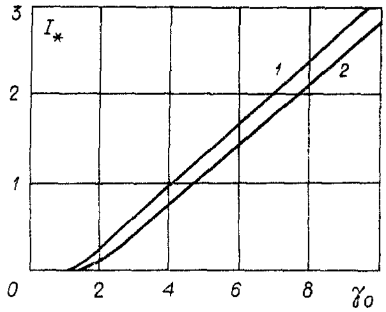

Fig. 3

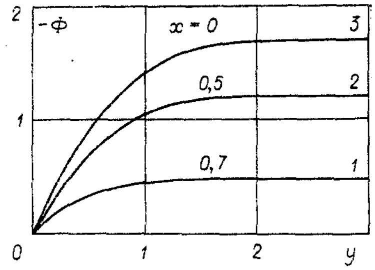

Fig. 4

The dependence of the limiting beam current on its energy was calculated by computer for $\alpha = 1$. Within the limits of accuracy of the calculation ($\sim 1\%$) this dependence agrees with Eq. (8).

The presence of the injection plane has practically no effect on the values of the attainable limiting vacuum current. This results from the fact that the potential in the system, as shown in Fig. 4, is smooth; no virtual cathode is observed for currents smaller than that determined by Eq. (8). Figure 4 also shows that at distances of the order of two waveguide radii from the injection plane the potential becomes the same as for an unbounded waveguide.

Thus, by investigating the dependence of the electrostatic potential distribution of a high-current electron beam on its energy in a two-dimensional transport system we have found the values of the limiting vacuum current. We have shown that taking account of the effect of the injection plane on the potential distribution has practically no effect on the values of the limiting current, since there are no singularities in the potential close to the injection plane.

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